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CLOSURE MODEL FOR INTERMITTENT TURBULENT FLOWS

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Abstract—A closure model for turbulent shear flows based exclusively on conditional moments and the intermittency factor is developed. The model contains the equations for intermittency factor, the turbulent zone and non-turbulent zone mean velocities and kinetic energy and dissipation rate for the turbulent zone. The model is applied to the prediction of plane jets and boundary layers and gives satisfactory results for intermittency factor and first order moments.

NOMENCLATURE

C_1, C_2, C_3, C_4, $C_D, C_{\varepsilon 1}, C_{\varepsilon 2}, C_{\varepsilon 3}$, C_{m1}, C_{k1},	turbulence model constants;
D_α,	flux of γ defined in equation (29);
$\mathscr{D}$,	domain;
$\mathscr{F}_\phi$,	point set defined by equation (3);
$\mathscr{F}_{\alpha\beta}$,	flux;
F, f,	interface term, function;
I,	indicator function;
k,	kinetic energy;
ln,	natural logarithm;
n_α,	normal vector of interface;
P,	probability density function;
p,	pressure;
Q,	source;
s,	interface expression defined in equation (3), source term;
t,	time;
$T_{\alpha\beta}$,	unitary transformation;
u, v, v_α,	velocity;
V,	relative progression velocity of interface;
x, x_α, y,	Cartesian coordinates;
Δ,	Laplace operator;
$\langle\ \rangle$,	ensemble average.

Greek symbols

δ,	Dirac function;
ε,	dissipation rate;
Φ,	scalar variable;
ϕ,	value of Φ;
Γ,	diffusivity;
γ,	intermittency factor;
λ,	Taylor scale;
Λ,	macro-scale;
ν,	kinematic viscosity;
ρ,	density;
σ,	Prandtl number;
ω_α,	vorticity.

Subscripts

k,	kinetic energy;
t,	turbulent;
α, β, γ,	coordinate directions;
γ,	intermittency factor;
ε,	dissipation rate;
ϕ,	scalar.

Superscripts

s,	interface;
$\bar{\ }$,	unconditional average;
$\tilde{\ }$,	turbulent zone average;
$\approx$,	non-turbulent zone average;
$*$,	turbulent zone fluctuation;
0,	non-turbulent zone fluctuation;
$'$,	unconditional fluctuation.

1. INTRODUCTION

TURBULENT shear flows exhibit an intermittent character in the neighbourhood of free boundaries. This was first established experimentally by Corrsin [1] and since then a large amount of experimental data for various shear flows has been gathered [2–9]. The theoretical treatment of intermittently turbulent flows however proved difficult in particular since certain aspects of it lead to the statistics of multi-valued random functions [10, 11]. The first attempts to establish closed equations for the intermittency factor and selected conditional moments are due to Libby [12, 13] who based his model however on a guessed transport equation for the intermittency factor. Dopazo [14] showed how to derive the exact equations for intermittency factor and conditional moments. After that several closure models have appeared in the literature [15, 16] which will be discussed in Section 2 of this paper.

It is well known that in turbulent shear flows of boundary layer type the intermittency factor follows closely a Gaussian distribution and there seems to be little reason to develop a closure model predicting it as

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solution of a complex system of partial differential equations. However, there are at least two classes of turbulent flows where the intermittent character is of primary importance and where a Gaussian distribution cannot be assumed: transitional flows and flows with fast chemical reactions. The closure model to be developed in this paper should provide a basis for an improved prediction of such flows but does not directly address them.

Finally it should be noted that any closure model of moment equations involves, in some form or other, assumptions concerning the probability density function (pdf) of fluctuating variables. If the fluctuations show an intermittent character the contribution from one of the zones ("non-turbulent") may produce a nearly-singular part of the pdf which in turn makes the pdf strongly non-Gaussian and any assumption based on quasi-Gaussian behaviour will fail. Conditioning of the variables allows the removal of such spikes in the pdf and consequently are closure assumptions based on quasi-Gaussianity better suited for conditional variables.

2. CONDITIONAL MOMENTS AND INTERMITTENCY

The description of the intermittent structure of turbulent flows requires several definitions based on the notion of a detector or discriminator variable. These quantities will be discussed first. Constant density will be assumed unless stated otherwise.

2.1. *Discriminating scalars and interfaces*

Consider a fluctuating scalar $\Phi(\mathbf{x}, t)$ which satisfies instantaneously the transport equation

$$\rho\frac{\partial\Phi}{\partial t} + \rho v_\alpha\frac{\partial\Phi}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(\rho\Gamma\frac{\partial\Phi}{\partial x_\alpha}\right) + \rho Q(\Phi) \quad (1)$$

and which is non-negative. In order to serve as discriminator Φ should satisfy two conditions:

(i) Φ should be below a small (compared to a turbulent reference value) threshold value in the non-turbulent region of the flow and be well above the threshold in the turbulent zone.

(ii) Φ should grow with the turbulent zone.

By stating these conditions it was implicitly assumed that the turbulent "state" of the flow could be defined uniquely. This is by no means straightforward (Lumley and Panofsky, [17]) but for the present purpose the properties random and vortical are used to define the turbulent state. This definition of turbulence leads to the choice of discriminator enstrophy:

$$\Phi(\mathbf{x}, t) \equiv \omega'_\alpha\omega'_\alpha, \quad (2)$$

or any monotonically non-decreasing function of it, where ω'_α is the fluctuating vorticity component. The measurement of vorticity is however quite difficult and therefore a class of scalars like temperature, colour, density are accepted as discriminators in suitable circumstances. This raises however the question whether both conditions stated above are satisfied for scalars different from (2). In particular for scalars that are instantaneously strictly conserved is no reason to believe that their growth (and hence the growth of the region where ϕ is above the threshold) will be the same as (2). Therefore we will use (2) in the following as discriminator.

Consider the solution $\Phi(\mathbf{x}, t)$ of (1) for given realization of the velocity field $v_\alpha(\mathbf{x}, t)$ and the boundary conditions for Φ. For differentiable v_α and sufficiently smooth boundary conditions the solution Φ will be differentiable and then the set of points $F_\phi(t)$ for which

$$S(\mathbf{x}, t; \phi) \equiv \Phi(\mathbf{x}, t) - \phi = 0, \quad (3)$$

and which can be approached as limit of points $\Phi(\mathbf{y}, t) < \phi$, is a surface in the flow domain D. If Φ is a discriminating scalar and if the threshold $\phi > 0$ is sufficiently small this surface can be regarded as mathematical model for the turbulent–non-turbulent interface. The real interface has a finite thickness which was estimated by Corrsin and Kistler [1] as being of the order of the Kolmogorov microscale. Only for the limit case of the turbulent Reynolds number going to infinity can we expect to encounter sharp (zero thickness) interfaces because the governing (instantaneous) equations allow only for this limit discontinuous (with respect to the discriminating scalar) solutions. The interface as defined above will therefore be contained within the physical interface and follow its movements.

2.2. *Conditional statistics*

Conditional statistics are a necessary tool for the description of intermittently turbulent flows. Only single point statistics will be considered here, and questions concerning geometrical properties of the interface such as distance from a coordinate plane leading to statistics of multi-valued functions (see [10, 11]) will be excluded. The interface will be assumed differentiable with probability one. Then consider the point $(\mathbf{x}, t)$ in the flow domain and define the indicator function $I(\mathbf{x}, t)$ as usual ([1, 14]) by

$$I(\mathbf{x}, t; \phi) \equiv \begin{cases} 1 & \text{if } S(\mathbf{x}, t) \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

where $\phi \geqslant 0$ is the threshold value of the discriminator Φ. The propagation velocity of the interface $F_\phi(t)$ is defined by

$$v^s_\alpha \equiv \frac{\mathrm{d}x^s_\alpha}{\mathrm{d}t}$$

where x^s_α is a point of the interface. Then the indicator function [18] follows, viz:

$$\frac{\partial I}{\partial t} + v_\alpha\frac{\partial I}{\partial x_\alpha} = V\delta(s) \quad (5)$$

where V is the relative progression velocity of the interface

$$Vn_\alpha = v_\alpha - v^s_\alpha \quad (6)$$

and n_α is the normal vector of $F_\phi(t)$ positive into the region with $I = 1$. The normal is defined and unique with probability one which does not exclude however the occurrence with zero probability of intersection points with several different values of n_α. The two basic rules for manipulating the indicator function [18] then follow:

$$\frac{\partial I}{\partial x_\alpha} = n_\alpha \delta(S) \tag{7}$$

and

$$\frac{\partial I}{\partial t} = -v_\alpha^s n_\alpha \delta(S) \tag{8}$$

which were already applied to derive equation (5). Denoting ensemble averages by angular brackets we can split any fluctuating quantity $\phi(\mathbf{x}, t)$ in three different ways:

$$\phi = \langle \phi \rangle + \phi' \tag{9}$$

where ϕ' is the unconditional fluctuation;

$$\phi = \bar{\bar{\phi}} + \phi^* \tag{10}$$

where $\bar{\bar{\phi}}$ denotes the turbulent zone average defined by

$$\bar{\bar{\phi}} \equiv \langle I\phi \rangle / \langle I \rangle; \tag{11}$$

and

$$\phi = \tilde{\tilde{\phi}} + \phi^0 \tag{12}$$

where $\tilde{\tilde{\phi}}$ denotes the non-turbulent zone average defined by

$$\tilde{\tilde{\phi}} \equiv \langle (1 - I)\phi \rangle / (1 - \langle I \rangle). \tag{13}$$

The average $\langle I \rangle$ of the indicator function is the intermittency factor and denoted by γ. From [9] and [13] follow various useful relations between conditional variables which are summarized in the appendix.

2.3. *Conditional moment equations*

The exact moment equations required for the development of the closure model will be given without derivation. The intermittency equation and the equations for conditional mean velocity and conditional turbulent kinetic energy can be found in [12] and [14], with a discussion of their properties. The equations for the intermittency factor γ can be obtained by ensemble averaging equation (5) for the indicator function and the result can be given as

$$\frac{\partial \gamma}{\partial t} + \bar{v}_\alpha \frac{\partial \gamma}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}[\gamma(1-\gamma)(\tilde{\tilde{v}}_\alpha - \bar{\bar{v}}_\alpha)] + S_\gamma \tag{14}$$

with source term S_γ

$$S_\gamma \equiv \langle V\delta(S) \rangle. \tag{15}$$

This term may still include diffusive, productive and destructive contributions and represents the mean entrainment of non-turbulent fluid into the turbulent zone. Furthermore it is possible that S_γ depends on the choice of the discriminating scalar Φ in particular whether Φ is strictly conserved or not. The equation for the turbulent zone mean velocity follows from the moment equations by multiplication with I, averaging, and applying (AI–AII). The result can be cast in the form

$$\frac{\partial \bar{\bar{v}}_\alpha}{\partial t} + \bar{\bar{v}}_\beta \frac{\partial \bar{\bar{v}}_\alpha}{\partial x_\beta} = \frac{\partial}{\partial x_\beta}\left(\nu \frac{\partial \bar{\bar{v}}_\alpha}{\partial x_\beta} - \overline{\overline{v_\alpha^* v_\beta^*}}\right) - \frac{1}{\rho}\frac{\partial \bar{\bar{P}}}{\partial x_\alpha} + \bar{\bar{F}}_\alpha. \tag{16}$$

The term $\bar{\bar{F}}_\alpha$ includes all influences of the interface fluctuations on the conditional mean velocity and consists of

$$\begin{aligned}\bar{\bar{F}}_\alpha \equiv{}& (1-\gamma)(\tilde{\tilde{v}}_\beta - \bar{\bar{v}}_\beta)\frac{\partial \bar{\bar{v}}_\alpha}{\partial x_\beta} - \overline{\overline{v_\alpha^* v_\beta^*}}\frac{\partial}{\partial x_\beta}\ln\gamma \\ &+ \frac{1}{\gamma}\left[-\nu\frac{\partial}{\partial x_\beta}\langle v_\alpha^* n_\beta \delta(S)\rangle \right.\\ &\left. + \langle (v_\alpha^* V + p^* n_\alpha)\delta(S)\rangle - \nu\left\langle \frac{\partial v_\alpha^*}{\partial x_\beta} n_\beta \delta(S)\right\rangle\right]\end{aligned} \tag{17}$$

For the closure model and in order to gain a better understanding of the terms in equation (17) the equation for the non-turbulent zone mean velocity $\tilde{\tilde{v}}_\alpha$ shall be given too. For the case that the non-turbulent zone has non-zero vorticity but the fluctuations v_α^0 have zero vorticity this equation follows as

$$\frac{\partial \tilde{\tilde{v}}_\alpha}{\partial t} + \tilde{\tilde{v}}_\beta \frac{\partial \tilde{\tilde{v}}_\alpha}{\partial x_\beta} = \frac{\partial}{\partial x_\beta}\left(\nu \frac{\partial \tilde{\tilde{v}}_\alpha}{\partial x_\beta} - \widetilde{\widetilde{v_\alpha^0 v_\beta^0}}\right) - \frac{1}{\rho}\frac{\partial \tilde{\tilde{P}}}{\partial x_\alpha} + \tilde{\tilde{F}}_\alpha, \tag{18}$$

where

$$\begin{aligned}\tilde{\tilde{F}}_\alpha \equiv{}& -\gamma(\bar{\bar{v}}_\beta - \tilde{\tilde{v}}_\beta)\frac{\partial \tilde{\tilde{v}}_\alpha}{\partial x_\beta} - \widetilde{\widetilde{v_\alpha^0 v_\beta^0}}\frac{\partial}{\partial x_\beta}\ln(1-\gamma) \\ &- \frac{1}{1-\gamma}\left[-\nu\frac{\partial}{\partial x_\beta}\langle v_\alpha^0 n_\beta \delta(S)\rangle \right.\\ &\left. + \langle (v_\alpha^0 V + p^0 n_\alpha)\delta(S)\rangle - \nu\left\langle \frac{\partial v_\alpha^0}{\partial x_\beta} n_\beta \delta(s)\right\rangle\right]\end{aligned} \tag{19}$$

and $\tilde{\tilde{k}} \equiv \frac{1}{2}\widetilde{\widetilde{v_\alpha^0 v_\alpha^0}}$ denotes the kinetic energy of the irrotational fluctuations. The first terms in equations (17) and (19) have opposite signs if $\partial\tilde{\tilde{v}}_\alpha/\partial x\beta$ and $\partial\bar{\bar{v}}_\alpha/\partial x_\beta$ have the same sign as can be expected in shear layers (jets, boundary layers) and then these terms represent a momentum exchange between turbulent and non-turbulent zones. The second term may have the same sign for both zones. In the high shear region of boundary layer flows the spatial derivative of γ is proportional to the crossing frequency of the interface and then the second terms in equations (17) and (19) can be interpreted as momentum sources due to the interface motion. Note that the second term in (17) is a momentum sink if the shear layer is a momentum defect flow (boundary layer, wake) and a momentum source if it is a momentum excess flow (jet). Finally, equations (17) and (19) contain various point-statistical correlations which have the same structure but opposite signs. These can be interpreted [14] as momentum flux through the interface.

The condition of zero vorticity for the fluctuations

v_α^0 has not been introduced into (19) but leads to the independent Corrsin–Kistler [14] equations:

$$\frac{\partial \widetilde{v_\alpha^0 v_\beta^0}}{\partial x_\beta} - \frac{\partial \tilde{k}}{\partial x_\alpha} = -\widetilde{v_\alpha^0 v_\beta^0}\frac{\partial}{\partial x_\beta}\ln(1-\gamma)$$

$$+ \tilde{k}\frac{\partial}{\partial x_\alpha}\ln(1-\gamma) + \frac{1}{1-\gamma}\langle (k^0 n_\alpha - v_\alpha^0 v_\beta^0 n_\beta)\delta(S)\rangle. \quad (20)$$

The transport equation for the turbulent zone Reynolds-stress tensor $\widetilde{v_\alpha^* v_\beta^*}$ can be given in the form

$$\frac{\partial}{\partial t}\widetilde{v_\alpha^* v_\beta^*} + \bar{v}_\gamma \frac{\partial}{\partial x_\gamma}\widetilde{v_\alpha^* v_\beta^*}$$

$$= \frac{\partial}{\partial x_\gamma}\left[\nu \frac{\partial}{\partial x_\gamma}\widetilde{v_\alpha^* v_\beta^*} - \widetilde{v_\gamma^* v_\alpha^* v_\beta^*} - \delta_{\beta\gamma}\widetilde{v_\alpha^* p^*} - \delta_{\alpha\gamma}\widetilde{v_\beta^* p^*}\right]$$

$$- \widetilde{v_\alpha^* v_\gamma^*}\frac{\partial \bar{v}_\beta}{\partial x_\gamma} - \widetilde{v_\beta^* v_\gamma^*}\frac{\partial \bar{v}_\alpha}{\partial x_\gamma}$$

$$+ \widetilde{p^*\left(\frac{\partial v_\alpha^*}{\partial x_\beta} + \frac{\partial v_\beta^*}{\partial x_\alpha}\right)} - \bar{\varepsilon}_{\alpha\beta} + F_{\alpha\beta}. \quad (21)$$

The symbol $\bar{\varepsilon}_{\alpha\beta}$ denotes the turbulent zone dissipation rate tensor defined by

$$\bar{\varepsilon}_{\alpha\beta} \equiv 2\nu \widetilde{\frac{\partial v_\alpha^*}{\partial x_\gamma}\frac{\partial v_\beta^*}{\partial x_\gamma}} \quad (22)$$

and $F_{\alpha\beta}$ represents the collected interface terms:

$$F_{\alpha\beta} \equiv (1-\gamma)(\bar{\bar{v}}_\gamma - \bar{v}_\gamma)\frac{\partial}{\partial x_\gamma}\widetilde{v_\alpha^* v_\beta^*}$$

$$+ \left[2\nu \frac{\partial}{\partial x_\gamma}\widetilde{v_\alpha^* v_\beta^*} - \widetilde{v_\gamma^* v_\alpha^* v_\beta^*} - \delta_{\beta\gamma}\widetilde{v_\alpha^* p^*} - \delta_{\alpha\gamma}\widetilde{v_\beta^* p^*}\right]$$

$$\times \frac{\partial}{\partial x_\gamma}\ln\gamma + \widetilde{v_\alpha^* v_\beta^*}\frac{1}{\gamma}(\nu\Delta\gamma - S_\gamma)$$

$$+ \frac{1}{\gamma}\Big[\langle v_\alpha^* v_\beta^* V\delta(S)\rangle + \langle (v_\alpha^* n_\beta + v_\beta^* n_\alpha)p^*\delta(S)\rangle$$

$$- \nu\left\langle n_\gamma \frac{\partial}{\partial x_\gamma}(v_\alpha^* v_\beta^*)\delta(S)\right\rangle - \nu\frac{\partial}{\partial x_\gamma}\langle v_\alpha^* v_\beta^* n_\gamma \delta(S)\rangle\Big], \quad (23)$$

with S_γ given by (15). Contraction of indices leads to the equation for the conditional kinetic energy of turbulence $\tilde{k}$

$$\tilde{k} \equiv \tfrac{1}{2}\widetilde{v_\alpha^* v_\alpha^*} \quad (24)$$

in the form (with $\bar{\varepsilon} = \frac{1}{2}\bar{\varepsilon}_{\alpha\alpha}$)

$$\frac{\partial \tilde{k}}{\partial t} + \bar{v}_\alpha \frac{\partial \tilde{k}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left[\nu\frac{\partial \tilde{k}}{\partial x_\alpha} - \widetilde{v_\alpha^* k^*} - \widetilde{v_\alpha^* p^*}\right]$$

$$- \widetilde{v_\alpha^* v_\beta^*}\frac{\partial \bar{v}_\alpha}{\partial x_\beta} - \bar{\varepsilon} + F_k. \quad (25)$$

The interface term $F_k = F_{\alpha\alpha}$ and is given by

$$F_k = (1-\gamma)(\bar{\bar{v}}_\alpha - \bar{v}_\alpha)\frac{\partial \tilde{k}}{\partial x_\alpha} + \left[2\nu\frac{\partial \tilde{k}}{\partial x_\alpha} - \widetilde{v_\alpha^* k^*} - \widetilde{v_\alpha^* p^*}\right]$$

$$\times \frac{\partial \ln\gamma}{\partial x_\alpha} + \tilde{k}\frac{1}{\gamma}(\nu\Delta\gamma - S_\gamma) + \frac{1}{\gamma}\Big[\langle k^* V\delta(S)\rangle$$

$$+ \langle v_\alpha^* n_\alpha p^*\delta(S)\rangle - \nu\left\langle n_\alpha\frac{\partial k^*}{\partial x_\alpha}\delta(S)\right\rangle$$

$$- \nu\frac{\partial}{\partial x_\alpha}\langle k^* n_\alpha \delta(S)\rangle\Big]. \quad (26)$$

It is worth noting that the redistributive correlation of pressure and rate of strain in (21) does not contribute to the balance of the conditional kinetic energy because the divergence

$$\frac{\partial v_\alpha^*}{\partial x_\alpha} = -\frac{\partial \bar{v}_\alpha}{\partial x_\alpha} \neq 0$$

but is non-fluctuating. Similarly follows the equation for the trace $\bar{\varepsilon}$ of the dissipation rate tensor using the relations given in the Appendix. It can be cast into the form

$$\frac{\partial \bar{\varepsilon}}{\partial t} + \bar{v}_\alpha\frac{\partial \bar{\varepsilon}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left[\nu\frac{\partial \bar{\varepsilon}}{\partial x_\alpha} - \widetilde{v_\alpha^* \varepsilon^*} - 2\nu\widetilde{\frac{\partial p^*}{\partial x_\beta}\frac{\partial v_\alpha^*}{\partial x_\beta}}\right]$$

$$- 2\nu\left(\widetilde{\frac{\partial v_\alpha^*}{\partial x_\gamma}\frac{\partial v_\beta^*}{\partial x_\gamma}} + \widetilde{\frac{\partial v_\gamma^*}{\partial x_\alpha}\frac{\partial v_\gamma^*}{\partial x_\beta}}\right)\frac{\partial \bar{v}_\alpha}{\partial x_\beta}$$

$$- 2\nu\widetilde{\frac{\partial v_\alpha^*}{\partial x_\beta}\frac{\partial v_\alpha^*}{\partial x_\gamma}\frac{\partial v_\beta^*}{\partial x_\gamma}} - 2\nu^2\widetilde{\frac{\partial^2 v_\alpha^*}{\partial x_\beta \partial x_\gamma}\frac{\partial^2 v_\alpha^*}{\partial x_\beta \partial x_\gamma}}$$

$$- 2\nu\widetilde{v_\beta^*\frac{\partial v_\alpha^*}{\partial x_\gamma}}\cdot\frac{\partial^2 \bar{v}_\alpha}{\partial x_\beta \partial x_\gamma} + F_\varepsilon \quad (27)$$

and the interface term is given by

$$F_\varepsilon = (1-\gamma)(\bar{\bar{v}}_\alpha - \bar{v}_\alpha)\frac{\partial \bar{\varepsilon}}{\partial x_\alpha} + \Big[2\nu\frac{\partial \bar{\varepsilon}}{\partial x_\alpha} - \widetilde{v_\alpha^* \varepsilon^*}$$

$$- 2\nu\widetilde{\frac{\partial p^*}{\partial x_\beta}\frac{\partial v_\alpha^*}{\partial x_\beta}}\Big]\frac{\partial \ln\gamma}{\partial x_\alpha} + \bar{\varepsilon}\frac{1}{\gamma}(\nu\Delta\gamma - S_\gamma)$$

$$+ \frac{1}{\gamma}\Big[\langle \varepsilon^* V\delta(S)\rangle + 2\nu\left\langle\frac{\partial p^*}{\partial x_\beta}\frac{\partial v_\alpha^*}{\partial x_\beta}n_\alpha\delta(S)\right\rangle$$

$$- \nu\left\langle n_\alpha\frac{\partial \varepsilon^*}{\partial x_\alpha}\delta(S)\right\rangle - \nu\frac{\partial}{\partial x_\alpha}\langle \varepsilon^* n_\alpha\delta(S)\rangle$$

$$- 2\nu\frac{\partial \bar{v}_\alpha}{\partial x_\alpha}\langle p^* n_\alpha \delta(S)\rangle\Big]. \quad (28)$$

The interface terms F_k and F_ε have the same structure as (26) and (28) show. The first term is as for

(16) a consequence of the introduction of the unconditional mean velocity in the convective part in (25) and (27). The second term is the product of the turbulent flux of $\bar{\tilde{k}}$ or $\bar{\tilde{\varepsilon}}$ respectively, times the gradient of the intermittency factor. The third term represents a reduction of $\bar{\tilde{k}}$ and $\bar{\tilde{\varepsilon}}$ due to the entrainment of non-turbulent fluid. The last term group contains the production/destruction of $\bar{\tilde{k}}$ and $\bar{\tilde{\varepsilon}}$ at the interface and the transport through it.

3. CLOSURE MODEL

A closure model on the basis of the $k - \varepsilon$ model initiated by Jones and Launder [19] will be developed for incompressible flows. The complete model will consist of a set of modelled equations for the intermittency factor plus turbulent zone mean velocity, kinetic energy, dissipation rate and non-turbulent zone mean velocity. Unconditional qualities such as mean velocity can be computed with the formulae given in the appendix.

3.1. *Intermittency equation*

The intermittency equation (14) contains two terms that require closure assumptions. The first is the transport of γ due to the relative motion of turbulent and non-turbulent zones:

$$D_\alpha \equiv -\gamma(1-\gamma)(\bar{\tilde{v}}_\alpha - \bar{\bar{v}}_\alpha). \quad (29)$$

This term could be considered closed if the crossflow component of the mean velocities was included in the model. For boundary layer type flows however the determination of these components becomes very inaccurate and therefore a model expression for this term is proposed. For boundary layer type shear flows the expression

$$-\gamma(1-\gamma) \sim \partial\gamma/\partial y$$

holds, where y is the crossflow coordinate. The coefficient of this derivative should then have the dimension of a (turbulent zone) viscosity and thus

$$D_\alpha \cong \bar{\tilde{v}}_t \cdot F_1(\gamma)\frac{\partial\gamma}{\partial x_\alpha}$$

with $F(\gamma)$ still to be determined. This expression would however contradict the experimental evidence in parabolic flows [3, 7, 8] where $|\bar{\tilde{u}} - \bar{\bar{u}}| > |\bar{\tilde{v}} - \bar{\bar{v}}|$. The reason for it is that an important contribution to the flux of γ due to the inhomogeneity of the mean velocity has not been included (see Lumley [15]). The complete model should be:

$$D_\alpha \cong F_1(\gamma)\bar{\tilde{v}}_t\frac{\partial\gamma}{\partial x_\alpha} - F_2(\gamma)\frac{\bar{\tilde{k}}}{\bar{\tilde{\varepsilon}}}(\bar{\tilde{v}}_\beta - \bar{\bar{v}}_\beta)\frac{\partial\bar{\tilde{v}}_\alpha}{\partial x_\beta}.$$

For parabolic flows the velocity difference in longitudinal direction is not required and for the crossflow component the second part can be neglected. Therefore, only $F_1(\gamma)$ has to be determined. The limits of $(\bar{\tilde{v}}_\alpha - \bar{\bar{v}}_\alpha)$ for $\gamma \to 0$ and $\gamma \to 1$ indicate what $F_1(\gamma)$ should be. From the fact that $|\bar{v}_\alpha - \bar{\tilde{v}}_\alpha|$ does not necessarily approach zero as $\gamma \to 0$ whereas [3] $|\bar{v}_\alpha - \bar{\bar{v}}_\alpha| \to 0$ as $\gamma \to 1$, it follows that D_α should approach zero faster for $\gamma \to 1$ than for $\gamma \to 0$. Hence for the crossflow-component of D_α the following are suggested:

$$F_1(\gamma) = 1 - \gamma$$

and

$$D_\alpha \cong (1-\gamma)\frac{\bar{\tilde{v}}_t}{\sigma_\gamma}\frac{\partial\gamma}{\partial x_\alpha} - F_2(\gamma)\frac{\bar{\tilde{k}}}{\bar{\tilde{\varepsilon}}}(\bar{\tilde{v}}_\beta - \bar{\bar{v}}_\beta)\frac{\partial\bar{\tilde{v}}_\alpha}{\partial x_\beta}\cdot \quad (30)$$

For complex turbulent flows however the equations for all mean velocity components will be solved and then (30) can be avoided.

The modelling of the intermittency source S_γ requires more detailed considerations. Several models have been proposed so far. Libby [12] proposed the expression

$$S_\gamma \cong \left(\frac{|\overline{u'v'}|}{U^2}\right)^{3/4}(1-\gamma)\frac{\bar{\tilde{U}} -}{\Lambda},$$

where U is a reference velocity and Λ is a length scale of the order of the width of the shear flow. Chevray and Tutu [9] suggested a modification of Libby's model

$$S_\gamma \cong 2C_3\gamma(1-\gamma)\frac{\bar{\tilde{U}}}{\Lambda}$$

with $C_3 = 3.447$, and the length scale $\Lambda = r_{1/2}$ the half-width of the round free jet which they studied experimentally. In the analysis of the conditional pdf of a passive scalar Γ in a non-buoyant turbulent plume O'Brien [16] suggested the model

$$S_\gamma \cong C\frac{D}{\lambda_s^2}\gamma\langle\Gamma\rangle P(0^+)$$

where D is the molecular diffusivity of Γ, λ_s is a turbulent micro-length scale for Γ, and $P(0^+)$ denotes the limit from $\Gamma > 0$ of the turbulent zone pdf P of the Γ-fluctuations. Libby's model produced good agreement between calculated and measured intermittency factor and conditional mean velocity. It was applied however only to self-similar flows and unconditional flow variables were prescribed [12]. Furthermore is this model neither Galilei-invariant nor applicable to complex flows where $\bar{\tilde{U}}$ might change sign? O'Brien's model is of limited use also, since the value of $P(\Gamma)$ approaching zero from $\Gamma > 0$ might not be bounded as the example beta-function

$$P(\Gamma) = \frac{\Gamma^{\alpha-1}(1-\Gamma)^{\beta-1}}{\int_0^1 \mathrm{d}\Gamma\Gamma^{\alpha-1}(1-\Gamma)^{\beta-1}} \qquad 0 \leqslant \Gamma \leqslant 1$$

shows, which leads to $P(0^+) \to \infty$ for $0 < \alpha < 1$ without an atomic contribution to the unconditional pdf at $\Gamma = 0$. Furthermore, it would require modelling of $P(0^+)$ in a set of equations not containing the pdf. The model suggested here consists of three contributions:

$$S_\gamma \cong S_\gamma^{(1)} + S_\gamma^{(2)} - S_\gamma^{(3)}, \quad (31)$$

and each contribution will be discussed in turn.

The first part of S_γ is

$$S_\gamma^{(1)} \cong -F_{(\gamma)}^{(1)} \cdot \frac{\overline{v_\alpha^* v_\beta^*}}{\bar{\bar{k}}} \cdot \frac{\partial \bar{\bar{v}}_\alpha}{\partial x_\beta} \quad (32)$$

and is based on the following considerations. The production of γ should be proportional to the production of kinetic energy of turbulence because an increase of $\bar{\bar{k}}$ implies an increase of vorticity fluctuations and therefore a production of the discriminator (2). This production of γ will however depend on γ itself, because the term (32) must not violate the realizability condition for γ, viz. $0 \leqslant \gamma \leqslant 1$. This leads to $F_{(\gamma)}^{(1)} = C \cdot \gamma(1-\gamma)$ in 1st-order. For turbulence models based on the turbulent viscosity concept the resulting form is for parabolic flows (x_1-dominant flow direction)

$$S_\gamma^{(1)} \cong C_1\gamma(1-\gamma)\frac{\bar{\bar{k}}}{\bar{\bar{\varepsilon}}}\left(\frac{\partial \bar{\bar{u}}}{\partial x_2}\right)^2. \quad (33)$$

The second part of S_γ is:

$$S_\gamma^{(2)} \cong C_2 F_{(\gamma)}^{(2)} \cdot \frac{\bar{\bar{k}}^2}{\bar{\bar{\varepsilon}}} \frac{\partial \gamma}{\partial x_\alpha} \frac{\partial \gamma}{\partial x_\alpha}. \quad (34)$$

The production of γ should be proportional to the gradient of γ. In fully developed shear flow turbulence a rapid variation of γ implies slow movement of turbulent fluid in the direction of the gradient of γ but rapid orthogonal to it, hence strong anisotropy. Since anisotropy of the Reynolds-stresses is counteracted by a part of the pressure–strain rate correlations this process must be accounted for in the γ-equation too and is represented by (34). This term can be interpreted as a non-linear convection which implies that (34) will not violate the realizability condition for above γ as long as γ is at least twice continuously differentiable. Hence reduces $F_{(\gamma)}^{(2)}$ in lowest order to a constant.

The third term $S_\gamma^{(3)}$ is:

$$S_\gamma^{(3)} = C_3 F_{(\gamma)}^{(3)} \frac{\bar{\bar{\varepsilon}}}{\bar{\bar{k}}} \quad (35)$$

Consider a positive threshold value γ of the discriminator defining the interface (3) and a homogeneous turbulence which is not maintained by external forces or boundary conditions. If initially $\gamma = 1$ then the equation for γ is

$$\frac{D\gamma}{Dt} = -C_3 F_{(\gamma)}^{(3)} \frac{\bar{\bar{\varepsilon}}}{\bar{\bar{k}}}$$

Realizability requires $F^{(3)}$ to be proportional to γ and in the limit $\phi \to 0$ to be proportional to $1-\gamma$ because $\Phi = 0$ would not be reached within a finite time interval. But for non-zero threshold values the discrimination will slip under the threshold within a finite time interval. Therefore the function $F_{(\gamma)}^{(3)}$ is $F_{(\gamma)}^{(3)} = \gamma(1-\gamma+\Delta)$, where Δ depends on ϕ. For developing shear layers and non-homogeneous flows however the influence of Δ becomes negligible except far downstream and for the calculations of these flows Δ was set zero. The final result for the intermittency equation is then for parabolic plane flows

$$\bar{U}\frac{\partial \gamma}{\partial x} + \bar{V}\frac{\partial \gamma}{\partial y} = \frac{\partial}{\partial y}\left[(1-\gamma)\frac{\bar{\bar{v}}_t}{\sigma_\gamma}\frac{\partial \gamma}{\partial y}\right] + C_1\gamma(1-\gamma)\frac{\bar{\bar{k}}}{\bar{\bar{\varepsilon}}}\left(\frac{\partial \bar{\bar{u}}}{\partial y}\right)^2 + C_2\frac{\bar{\bar{k}}^2}{\bar{\bar{\varepsilon}}}\left(\frac{\partial \gamma}{\partial y}\right)^2 - C_3\gamma(1-\gamma)\frac{\bar{\bar{\varepsilon}}}{\bar{\bar{k}}} \quad (36)$$

The model constants are summarized in Table 1.

3.2. *Conditional moment equations*

The closure of the turbulent zone momentum equation (16) and the equations for the turbulent zone kinetic energy (25) and its dissipation rate (27) is based on the concept of turbulent viscosity as applied by Jones and Launder to the k–ε model [19]. In fact the same closure assumptions will be employed but for turbulent zone moments only. For the momentum equation (16), in the dominant flow direction x two closure assumptions are required. The turbulent zone shear stress is expressed as

$$-\widetilde{u^* v^*} \cong \bar{\bar{v}}_t \frac{\partial \bar{\bar{u}}}{\partial y}. \quad (37)$$

analogous to the unconditional model (but not a consequence of it). The second term F_γ in (16) however does not have a counterpart in the unconditional equation and the last part of (17) requires therefore a new closure assumption. The interface terms in (17) are modelled by

$$\langle(v_\alpha^* V + p^* n_\alpha)\delta(S)\rangle - \nu\frac{\partial}{\partial x_\beta}\langle v_\alpha^* n_\beta \delta(S)\rangle - \nu\left\langle\frac{\partial v_\alpha^*}{\partial x_\beta} n_\beta \delta(S)\right\rangle \cong C \cdot \bar{\bar{v}}_t\left(\frac{\partial \bar{\bar{v}}_\alpha}{\partial x_\beta} + \frac{\partial \bar{\bar{v}}_\beta}{\partial x_\alpha}\right)\frac{\partial \gamma}{\partial x_\beta} \quad (38)$$

and with (30), (37) the model for (17) after rearrangement is then

$$F_1 \cong \frac{\bar{\bar{v}}_t}{\sigma_\gamma}\frac{\partial \ln \gamma}{\partial y}\frac{\partial \bar{\bar{u}}}{\partial y} \cdot (C_{m1} - \gamma). \quad (39)$$

The form (38) suggested for the interface terms is based on the assumption that they are dominated by the mean strain rate in the turbulent zone and the gradient of the intermittency factor as a measure for the crossing frequency. The resulting model (39) acts in shear layers as a momentum sink if the non-turbulent flow is faster and as source if the turbulent flow is faster and therefore increases the spreading rate of the turbulent zone. The resulting closed equation for the turbulent zone mean velocity $\bar{\bar{u}}$ is then

$$\bar{\bar{u}}\frac{\partial \bar{\bar{u}}}{\partial x} + \bar{\bar{v}}\frac{\partial \bar{\bar{u}}}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \bar{\bar{v}}_t)\frac{\partial \bar{\bar{u}}}{\partial y}\right] - \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{\bar{\bar{v}}_t}{\sigma_\gamma}\frac{1}{\gamma}\frac{\partial \gamma}{\partial y}\frac{\partial \bar{\bar{u}}}{\partial y}(C_{m1} - \gamma). \quad (40)$$

The equations for the turbulent zone kinetic energy $\bar{\bar{k}}$ and its dissipation rate require one additional modelling assumption. The model for F_k follows from (30), (37) and the expression

$$\langle k^* V\delta(S)\rangle + \langle v_\alpha^* p^* n_\alpha \delta(S)\rangle - \nu \left\langle n_\alpha \frac{\partial k^*}{\partial x_\alpha}\delta(S)\right\rangle - \nu\frac{\partial}{\partial x_\alpha}\langle k^* n_\alpha \delta(S)\rangle \cong C_{k1}\gamma(1-\gamma)\bar{\bar{\varepsilon}}$$

for the interface terms as

$$F_k \cong \left(\frac{1}{\sigma_k} + \frac{1-\gamma}{\sigma_\gamma}\right)\bar{\bar{\nu}}_t \frac{1}{\gamma}\frac{\partial \gamma}{\partial y}\frac{\partial \bar{\bar{k}}}{\partial y} + \frac{1}{\gamma}\left(\nu\frac{\partial^2\gamma}{\partial y^2} - S_\gamma\right)\bar{\bar{k}} + C_{k1}(1-\gamma)\bar{\bar{\varepsilon}} \quad (41)$$

where S_γ is given by (31)–(35).

In the same manner the model for F_ε is obtained as

$$F_\varepsilon \cong \left(\frac{1}{\sigma_\varepsilon} + \frac{1-\gamma}{\sigma_\gamma}\right)\bar{\bar{\nu}}_t \frac{1}{\gamma}\frac{\partial \gamma}{\partial y}\frac{\partial \bar{\bar{k}}}{\partial y} + \frac{1}{\gamma}\left(\nu\frac{\partial^2\gamma}{\partial y^2} - S_\gamma\right)\bar{\bar{\varepsilon}} + C_{\varepsilon 3}(1-\gamma)\frac{\bar{\bar{\varepsilon}}^2}{\bar{\bar{k}}}. \quad (42)$$

With these two expressions the closed equations for the turbulent zone kinetic energy and its dissipation rate follow as:

$$\bar{\bar{u}}\frac{\partial \bar{\bar{k}}}{\partial x} + \bar{\bar{v}}\frac{\partial \bar{\bar{k}}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\bar{\bar{\nu}}_t}{\sigma_k}\right)\frac{\partial \bar{\bar{k}}}{\partial y}\right] + \bar{\bar{\nu}}_t\left(\frac{\partial \bar{\bar{u}}}{\partial y}\right)^2 - \bar{\bar{\varepsilon}} + F_k \quad (43)$$

and

$$\bar{\bar{u}}\frac{\partial \bar{\bar{\varepsilon}}}{\partial x} + \bar{\bar{v}}\frac{\partial \bar{\bar{\varepsilon}}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\bar{\bar{\nu}}_t}{\sigma_\varepsilon}\right)\frac{\partial \bar{\bar{\varepsilon}}}{\partial y}\right] + C_{\varepsilon 1}C_D\bar{\bar{k}}\left(\frac{\partial \bar{\bar{u}}}{\partial y}\right)^2 - C_{\varepsilon 2}\frac{\bar{\bar{\varepsilon}}^2}{\bar{\bar{k}}} + F_\varepsilon \quad (44)$$

The turbulent zone pseudo-viscosity is then analogously to the unconditional case defined by

$$\bar{\bar{\nu}}_t = C_D\frac{\bar{\bar{k}}^2}{\bar{\bar{\varepsilon}}}. \quad (45)$$

The closed equations for turbulent zone mean velocity (40), kinetic energy (43) and dissipation rate (44) reduce for γ approaching unity ro the unconditional model if the first and second derivatives approach zero too. Experiments show that this can indeed be expected for fully developed shear flows [3, 7], in which the γ-profile reaches a nearly-constant (unity) region in the centre part of the flow.

The turbulent zone quantities however are not sufficient to determine unconditional moments as (A1) and (A7) prove. Therefore, the closure of equation (18) for the non-turbulent zone velocity is required for the calculation of the unconditional mean velocity.

For the closure of equation (18) the stress tensor in the non-turbulent zone and the interface terms F_α must be estimated. The experimental evidence in parabolic flows [3, 7, 8] shows that the stress tensor cannot be neglected immediately because the normal stress components in the non-turbulent zone can amount to 40% of their turbulent zone counterparts. The shear stress in the non-turbulent zone however is much smaller than the turbulent zone shear stress and could be neglected. From this follows that no simple relation between the stress tensors in turbulent and non-turbulent zone can hold component-wise but possibly in more general tensorial form via the unitary transform $T_{\alpha\gamma}$ corresponding to a rotation of the local coordinate system

$$\widetilde{\overline{v_\alpha^0 v_\beta^0}} \cong \text{const.}\, f(\gamma)\, T_{\alpha\gamma}\, \widetilde{\overline{v_\gamma^* v_\beta^*}}\, T^t_{\delta\beta}. \quad (46)$$

The unitary transform $T_{\alpha\gamma}$ is determined by the angles between the principal axis of the stress tensors. For plane and axisymmetric flows the principal axis of the stress tensor in the non-turbulent zone is close to the gradient of the intermittency factor as the experiments show [7, 8]. Hence is this angle given by the principal axis of $\widetilde{\overline{v_\alpha^* v_\beta^*}}$ and the gradient of γ. For the kinetic energies, which are invariants of the two tensors, however follows the simpler relation

$$\tilde{k} = C_5\, f(\gamma)\bar{\bar{k}} \quad (47)$$

only requiring the knowledge of the constant and the function $f(\gamma)$. The data for boundary layers [7, 8] suggest for the constant a value of $C_5 \cong 0.5$ and for $f(\gamma) = \gamma$.

The modelling of equation (18) however follows from the fact that the shear stress in the non-turbulent zone is an order of magnitude of smaller than the shear stress in the turbulent zone that the corresponding term can be neglected. This leaves the interface term (19) to be modelled. This term cannot be negligible because following from $\tilde{F}_\alpha \cong 0$ together with $\widetilde{u^0 v^0} \cong 0$ is the existence of a laminar flow in the non-turbulent zone at Re numbers well in the turbulent regime which cannot be stable. Considering the order of magnitude of the terms constituting $\tilde{F}_\alpha$ the first two are negligible in contrast to (17) where the stress tensor has to be kept. The approximation of an inviscid flow is reasonable for the non-turbulent zone in most cases (except transitional flows), thus the viscous terms in (19) are negligible and $\tilde{F}_\alpha$ reduces to

$$\tilde{F} \cong -\frac{1}{1-\gamma}\langle (v_\alpha^0 V + p^0 n_\alpha)\delta(S)\rangle.$$

Before suggesting any closure expression for this term the boundary conditions for the non-turbulent zone velocity must be considered. For fully developed turbulent shear flows the probability for a patch of non-turbulent fluid to reach the opposite boundary becomes very small and therefore the limit $\gamma \to 1$ for non-turbulent zone quantities loses its meaning. However these quantities must still remain bounded for $\gamma \to 1$ if a boundary value problem for non-turbulent

Table 1. Model constants

C_1	0.2	σ_γ	1.0	$C_{\varepsilon 1}$	1.44
C_2	0.6	σ_k	1.0	$C_{\varepsilon 2}$	1.95
C_3	0.1	σ_ε	1.3	$C_{\varepsilon 3}$	2.5
C_4	0.5	C_{m1}	0.7	C_{k1}	2.5
C_D	0.09				

The values for $C_D, \sigma_k, \sigma_\varepsilon, C_{\varepsilon 1}, C_{\varepsilon 2}$ are standard values taken from the original k–ε model [19].

zone variables is to be solved. On the rare occasion of a non-turbulent patch of fluid penetrating to the other parts of the boundary it must satisfy the same conditions as the turbulent eddies. Hence the boundary conditions for the non-turbulent zone velocity should be the same as for the turbulent zone quantity.

To satisfy this condition the interface term must therefore diffuse the momentum from the boundaries into the flow field and create/destroy momentum there. Recalling the alternative way of representing the Dirac function defined on the surface $S(\mathbf{x}, t) = 0$ as introduced by Dopazo [14] we can write for the first part of $\bar{F}_\alpha$:

$$-\frac{1}{1-\gamma}\langle v_\alpha^0 V\delta(S)\rangle = \lim_{\delta\to 0}\left\langle\frac{1}{V(u_\delta)}\int_{U_\delta\cap S(t)} \mathrm{d}s\, v_\alpha V\right\rangle,$$

(where u_δ is a sphere of radius δ around $\mathbf{x}$ and $v(u_\delta)$ its volume) which can now be recognized as momentum flux across the interface per unit volume due to the fluctuations of the relative progression velocity V and represents therefore the divergence of a flux. Hence the model

$$-\left\langle\frac{v_\alpha^0 V}{1-\gamma}\delta(S)\right\rangle \cong -\frac{\partial \mathscr{F}_{\alpha\beta}}{\partial x_\beta} \tag{48}$$

with

$$-\mathscr{F}_{\alpha\beta} \cong C_4 f(\gamma)\,\bar{v}_t \frac{\partial \bar{\bar{v}}_\alpha}{\partial x_\beta} \tag{49}$$

is suggested. Since non-turbulent zone fluctuations decay rapidly with distance from the interface [20] the flux $\mathscr{F}_{\alpha\beta}$ will be proportional to γ and therefore $f(\gamma) = \gamma$. The second part of F_α containing the pressure fluctuations is neglected because p^0 is determined by large scale fluctuations whereas n_α corresponds to gradient fluctuations which are dependent upon small scale fluctuations whose correlations be expected to be weak.

The closed equation for $\bar{\bar{u}}$ follows now as

$$\bar{\bar{u}}\frac{\partial\bar{\bar{u}}}{\partial x} + \bar{\bar{v}}\frac{\partial\bar{\bar{u}}}{\partial y} = \frac{\partial}{\partial y}\left[(v + C_4\gamma\bar{v}_t)\frac{\partial\bar{\bar{u}}}{\partial y}\right] - \frac{1}{\rho}\frac{\partial\bar{\bar{p}}}{\partial x}. \tag{50}$$

The constants are summarized in Table 1.

4. APPLICATION

The modelled transport equations for the intermittency factor and the turbulent zone and non-turbulent zone moments are discretized and solved using standard finite-difference procedures [21]. The plane jet and the boundary layer on a smooth wall at zero pressure gradient are considered for comparison of calculation and experiment.

4.1. *Plane jet*

The measurements of Gutmark and Wygnanski [3] in the plane jet provide an excellent basis for comparison with the calculations. The boundary conditions for this case are zero gradient conditions at the symmetry axis and Dirichlet conditions at the outer edge of the flow domain. At the jet pipe exit rectangular velocity profiles $\tilde{u} = \bar{\bar{u}}$ and a small turbulence level were prescribed and the intermittency factor was set to $\gamma_0 = 0.001$ at this section. The intermittency factor develops downstream first in the plane shear layer until the core region of the jet has vanished and then adjusts to the profile to be expected in the jet as shown in Fig. 1. In the self-similar region (Fig. 2) the intermittency factor agrees well with the experimental distribution. The bars in Fig. 2 give the variation of the experimental results if the measurements of Heskestad [5] and Bradbury [4] are taken into account. For an ideal Gaussian distribution of $\gamma(y)$ and no folding of the interface, the cross-derivative of γ would (properly normalized) be the Gaussian probability density function of the interface position. The calculated $\partial\gamma/\partial y$ deviates from the measured crossing frequency considerably and shows that the calculated maximum is closed to $\gamma = 0.5$ than the measured maximum, indicating a stronger deviation from Gaussianity in the measurements. The unconditional and turbulent zone mean velocities in Fig. 3 show close agreement between measurement and calculation. The current

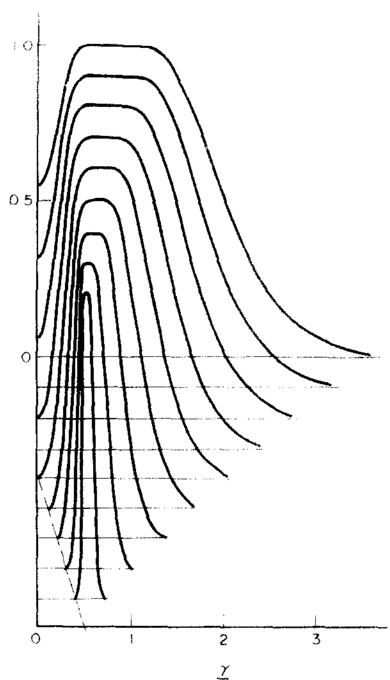

FIG. 1. Initial development of intermittency factor in plane jet over the range $0 \leqslant x/D \leqslant 10$.

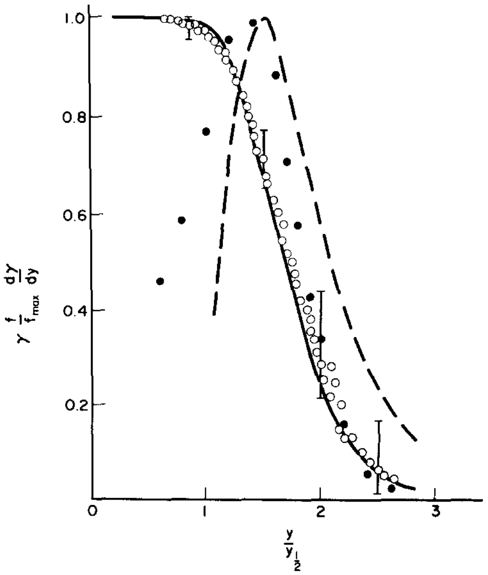

FIG. 2. Intermittency factor γ and $d\gamma/dy$ in self-similar region of plane jet compared with measurements [3].

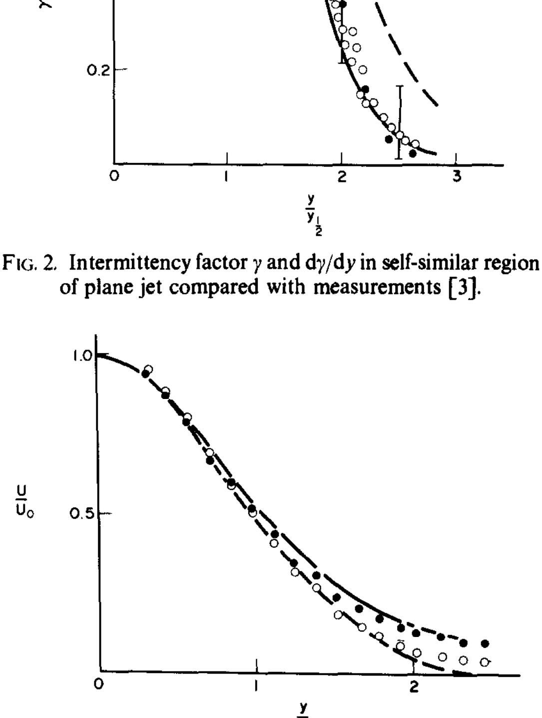

FIG. 3. Turbulent zone (solid line) and unconditional (broken line) mean velocities in plane jet compared with experiments [3] (symbols).

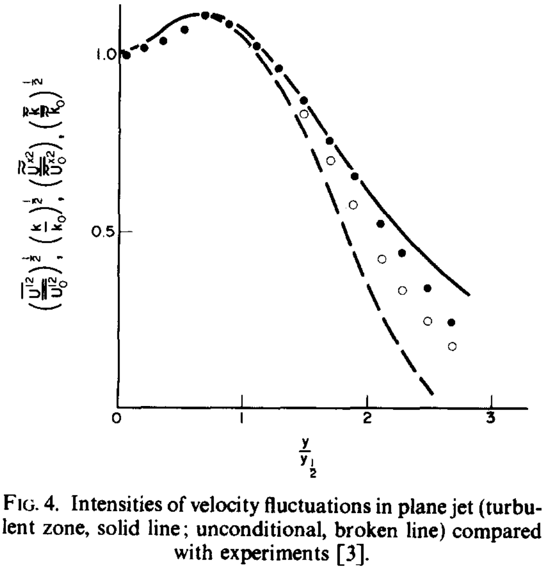

FIG. 4. Intensities of velocity fluctuations in plane jet (turbulent zone, solid line; unconditional, broken line) compared with experiments [3].

prediction model employing the concept of turbulent viscosity cannot be expected to give excellent results for 2nd-order moments. The fluctuation intensity of velocity in the turbulent zone and unconditional (Fig. 4) normalized with the respective axial values shows the correct distribution across the jet but their difference is overpredicted in the outer part.

4.2. *Boundary layer*

For the calculation of turbulent boundary layers low *Re* number corrections must be introduced in order to represent properly the region close to the wall where the turbulent *Re* number drops to zero. The correlations suggested by Jones and Launder [19] are applied to the turbulent zone variables without modification. The boundary conditions used in the calculations are Dirichlet conditions at both edges of the flow except for the intermittency factor which satisfies a zero gradient condition at the wall. The initial profiles are given by $\bar{\gamma} = 0.001$ and $\tilde{\bar{u}}(x_0, y)$ as laminar boundary layer and $\bar{\bar{u}}(x_0, y)$ as turbulent boundary layer profile. The initial boundary layer thicknesses are $\delta(\tilde{\bar{u}}) = 0.003$ m and $\delta(\bar{\bar{u}}) = 1.7\delta(\tilde{\bar{u}})$ and the initial profiles for $\bar{\bar{k}}$ and $\bar{\bar{\varepsilon}}$ correspond to a turbulent boundary layer. The unconditional distributions of velocity and kinetic energy correspond however at the beginning to a laminar boundary layer. The downstream development is dominated by a rapid growth of γ in the lower part of the boundary layer which in turn leads to a growth of the unconditional kinetic energy and a change of velocity profile to the turbulent form which is achieved over a distance of the same order of magnitude ($x - x_0 \sim 0.4$ m) as the transition region. After this initial region the development of the boundary layer is calculated until the Re_θ-number of the experiments of Kovasznay, Kibens and Blackwelder [7] is reached ($\delta = 0.09$ m). The results at this point are presented in Figs. 5–8. The intermittency factor agrees well with the measurements [7, 8] and the derivative $\partial\gamma/\partial y$ is much closer to the measured crossing frequency than in the jet (Fig. 2). The fluctuation intensities (Fig. 6) for turbulent zone, unconditional and non-turbulent zone velocity fluctuations agree reasonably with the measured $u'/\bar{u}_\infty$ profile. As for the jet, the turbulent zone fluctuations are larger than the measured values. The mean velocities in the outer part of the boundary layer (Fig. 7) show for all three statistics the same form as the experiment. Finally the model (47) for the non-turbulent zone fluctuations of v_α is compared in Fig. 8 with the measurements. According to Philips' hypothesis [22] $(\widetilde{u^{02}}/\bar{u}_\infty^2)^{-1/4}$ should vary linearly with y. The predictions are slightly curved but fairly close to the experimental values and can be considered satisfactory within the framework of a closure model based on the concept of a turbulent viscosity.

5. CONCLUSIONS

A closure model for turbulent shear flows based exclusively on conditional moments and the intermit-

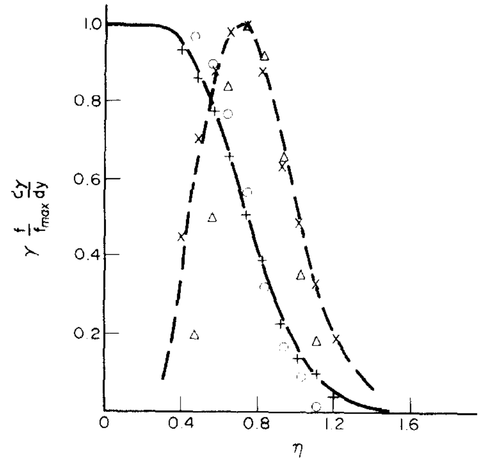

FIG. 5. Intermittency factor γ and $d\gamma/dy$ in turbulent boundary layer ($Re_\delta = 21{,}000$) compared with measurements (○ and △ [7]), + and × [8].

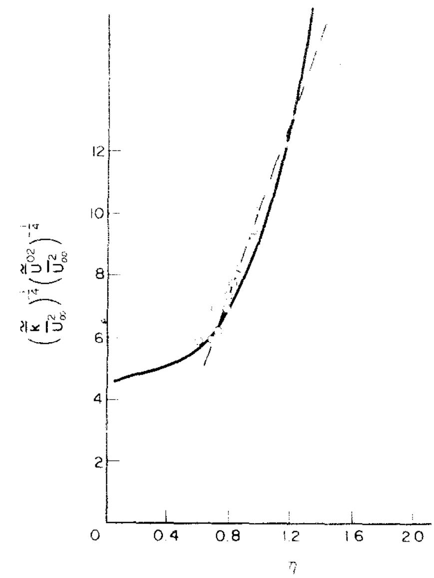

FIG. 8. Non-turbulent zone fluctuations compared with measurements [7] and Philips' hypothesis (broken line).

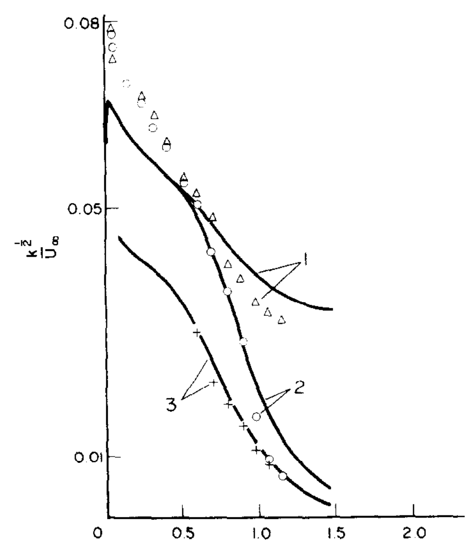

FIG. 6. Intensity of velocity fluctuations: (1) turbulent zone, (2) unconditional, (3) non-turbulent zone, compared with measurements [7] of u'/u_∞.

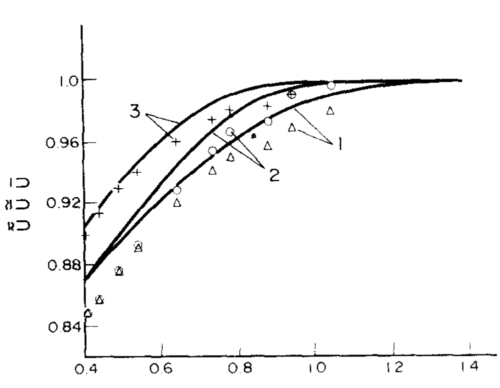

FIG. 7. Mean velocities in outer part of boundary layer: (1) turbulent zone, (2) unconditional (3) non-turbulent zone compared with measurements [7].

tency factor was developed. The model contains the equations for intermittency factor, the turbulent zone and non-turbulent zone mean velocities and kinetic energy and dissipation rate for the turbulent zone. The main part of the paper was devoted to the closure of the equation for the intermittency factor: production of γ via the kinetic energy created in the turbulent zone and the inhomogeneity of γ-field, destruction by viscous effects and turbulent diffusion. The conditional moment equations contain a new-term group representing the effect of moving interface, and models representing them were suggested. The results obtained with this model give satisfactory agreement for intermittency factor and conditional moments of first order. This shows the possibility of treating turbulent flows, in which transport processes across interfaces play an important role (such as turbulent flames) with conditional turbulence models. Such models would allow the transport process across the interface to be dealt with explicitly because there are terms in the equations accounting for them.

REFERENCES

1. S. Corrsin and A. L. Kistler, The free-stream boundaries of turbulent flows. NACA TN 3133 (1954).
2. H. Fiedler and M. R. Head, Intermittency measurements in the turbulent boundary layer. *J. Fluid Mech.* **25**, 719 (1966).
3. E. Gutmark and I. Wygnanski, The planar turbulent jet, *J. Fluid Mech.* **73**, 465 (1976).
4. L. J. S. Bradbury, The structure of a self-preserving turbulent plane jet, *J. Fluid Mech.* **23**, 31 (1975).
5. G. Heskestad, Hot-wire measurements in a plane turbulent jet, *Trans. ASME, Ser. E. J. appl. Mech.* **32**, 721 (1965).
6. S. T. Paizis and W. H. Schwarz, An investigation of the topography and motion of the turbulent interface, *J. Fluid Mech.* **63**, 315 (1974).
7. L. S. G. Kovasznay, V. Kibens and R. F. Blackwelder,

Large-scale motion in the intermittent region of a turbulent boundary layer, *J. Fluid Mech.* **41**, 283 (1970).
8. T. B. Hedley and J. F. Keffer, Some turbulent/non-turbulent properties of the outer intermittent region of a boundary layer. *J. Fluid Mech.* **64**, 645 (1974).
9. R. Chevray and N. K. Tutu, Intermittency and preferential transport of heat in a round jet, *J. Fluid Mech.* **88**, 133 (1978).
10. S. Corrsin and O. M. Philips, Contour length and surface area of multiple-valued random variables, *J. Soc. indust. appl. Math.* **9**, 395 (1961).
11. J. L. Lumley, On multiple-valued random functions, *J. math. Phys.* **5**, 1198 (1964).
12. P. A. Libby, On the prediction of intermittent turbulent flows, *J. Fluid Mech.* **68**, 273 (1975).
13. P. A. Libby, Prediction of the intermittent turbulent wake of a heated cylinder, *Phys. Fluids* **19**, 494 (1976).
14. C. Dopazo, On conditional averages for intermittent turbulent flows, *J. Fluid Mech.* **81**, 433 (1977).
15. J. L. Lumley, Second order modelling of turbulent flows, in *Prediction Methods for Turbulent Flows* (Edited by W. Kollmann), Vol. 1, Hemisphere (1980).
16. E. E. O'Brien, Stochastic properties of scalar quantities advected by a non-buoyant plume, *J. Fluid Mech.* **89**, 209 (1978).
17. J. L. Lumley and H. A. Panofsky, *The structure of atmospheric turbulence.* Interscience (1964).
18. I. M. Gelfand and G. E. Shilov, *Generalized functions.* Vol. 1. Academic Press (1964).
19. W. P. Jones and B. E. Launder, The prediction of laminarization with a two equation model of turbulence, *Int. J. Heat Mass Transfer* **15**, 2, 301 (1972).
20. A. A. Townsend, *The structure of turbulent shear flows.* (2nd ed.) Cambridge University Press (1976).
21. S. V. Patankar and D. B. Spalding, *Heat and mass transfer in boundary layers.* Intertext, London (1970).
22. O. M. Philips, The entrainment interface, *J. Fluid Mech.* **51**, 97 (1972).

APPENDIX

The conditional statistics introduced in Section 2.2 imply various relations between conditional and unconditional quantities. The mean values are related by

$$\langle\phi\rangle = \gamma\bar{\bar{\phi}} + (1-\gamma)\bar{\tilde{\phi}} \tag{A1}$$

as a consequence of (11) and (13). The proper mean values of the fluctuating parts are

$$\bar{\bar{\phi}}^* = \bar{\tilde{\phi}}^0 = 0,$$

but mean values with respect to different statistics are generally non-zero. In particular the following relations hold for mean values of fluctuating components. For the conditional fluctuations averaged unconditionally we obtain:

$$\langle\phi^*\rangle = (1-\gamma)(\bar{\bar{\phi}} - \bar{\tilde{\phi}}), \tag{A2}$$

$$\langle\phi^0\rangle = \gamma(\bar{\bar{\phi}} - \bar{\tilde{\phi}}), \tag{A3}$$

and for the unconditional fluctuation it follows that:

$$\bar{\bar{\phi}}' = -\langle\phi^*\rangle, \tag{A4}$$

$$\bar{\tilde{\phi}}' = -\langle\phi^0\rangle. \tag{A5}$$

Furthermore we note that

$$\phi' = \gamma\phi^* + (1-\gamma)\phi^0. \tag{A6}$$

The 2nd-order moments are given by

$$\langle\phi'\psi'\rangle = \gamma\overline{\widetilde{\phi^*\psi^*}} + (1-\gamma)\widetilde{\overline{\phi^0\psi^0}} + \gamma(1-\gamma)(\bar{\bar{\phi}} - \phi)(\bar{\bar{\psi}} - \bar{\tilde{\psi}}). \tag{A7}$$

This relation shows that the unconditional correlation is not necessarily between the conditional correlations but may be larger or smaller than $\overline{\widetilde{\phi^*\psi^*}}$ and $\widetilde{\overline{\phi^0\psi^0}}$. The differentiation rules (7) and (8) for the indicator function and (9)–(13) imply the following relations:

$$\overline{\widetilde{\frac{d\phi^*}{\partial x_\alpha}}} = -\frac{1}{\gamma}\langle\phi^* n_\alpha\delta(S)\rangle \tag{A8}$$

and

$$\overline{\widetilde{\frac{\partial\phi^*}{\partial t}}} = \frac{1}{\gamma}\langle\phi^* v_\alpha^s n_\alpha\delta(S)\rangle. \tag{A9}$$

Note that (A8) and (A9) imply that the point-statistical mean values of differentiated fluctuations do not necessarily vanish. Finally the correlations involving differentiation can be evaluated as

$$\overline{\widetilde{\phi\frac{\partial\psi}{\partial x_\alpha}}} = \frac{\partial}{\partial x_\alpha}\overline{\widetilde{\phi\psi}} - \overline{\widetilde{\psi\frac{\partial\phi}{\partial x_\alpha}}} + \overline{\widetilde{\phi\psi}}\frac{\partial}{\partial x_\alpha}\ln\gamma - \frac{1}{\gamma}\langle\phi\psi n_\alpha\delta(S)\rangle \tag{A10}$$

and

$$\overline{\widetilde{\phi\frac{\partial\psi}{\partial t}}} = \frac{\partial}{\partial t}\overline{\widetilde{\phi\psi}} - \overline{\widetilde{\psi\frac{\partial\phi}{\partial t}}} + \overline{\widetilde{\phi\psi}}\frac{\partial\ln\gamma}{\partial t} + \frac{1}{\gamma}\langle\phi\psi v_\alpha^s n_\alpha\delta(S)\rangle\cdot \tag{A11}$$

For non-turbulent zone quantities follows

$$\widetilde{\overline{\frac{\delta\phi^0}{\partial x_\alpha}}} = \frac{1}{1-\gamma}\langle\phi^0 n_\alpha\delta(S)\rangle \tag{A12}$$

and

$$\overline{\frac{\partial\phi^0}{\partial t}} = -\frac{1}{1-\gamma}\langle\phi^0 v_\alpha^s n_\alpha\delta(S)\rangle, \tag{A13}$$

and for correlations

$$\widetilde{\overline{\phi\frac{\partial\psi}{\partial x_\alpha}}} = \frac{\partial}{\partial x_\alpha}\widetilde{\overline{\phi\psi}} - \widetilde{\overline{\psi\frac{\partial\phi}{\partial x_\alpha}}} + \widetilde{\overline{\phi\psi}}\frac{\partial}{\partial x_\alpha}\ln(1-\gamma) + \frac{1}{1-\gamma}\langle\phi\psi n_\alpha\delta(S)\rangle \tag{A14}$$

and

$$\widetilde{\overline{\phi\frac{\partial\psi}{\partial t}}} = \frac{\partial}{\partial t}\widetilde{\overline{\phi\psi}} - \widetilde{\overline{\psi\frac{\partial\psi}{\partial t}}} + \widetilde{\overline{\phi\psi}}\frac{\partial}{\partial t}\ln(1-\gamma) - \frac{1}{1-\gamma}\langle\phi\psi v_\alpha^s n_\alpha\delta(S)\rangle. \tag{A15}$$

Specialization of ϕ and ψ can be used to produce relations for higher order moments. Finally we note a consequence of (A6) and the fact that the velocity is continuous across the interface for the point-statistical part of the momentum sources F_α. Denote by

$$S_\alpha \equiv \langle(v_\alpha V + pn_\alpha)\delta(S)\rangle - \nu\left\langle\frac{\partial V_\alpha}{\partial x_\beta}n_\beta\delta(S)\right\rangle - \nu\frac{\partial}{\partial x_\beta}\langle v_\alpha n_\beta\delta(S)\rangle,$$

then

$$S_\alpha^*\gamma + S_\alpha^0(1-\gamma) = S'_\alpha + 2_\nu(\bar{\bar{v}}_\alpha - \bar{\tilde{v}}_\alpha)\frac{\partial\gamma}{\partial x_\beta}\frac{\partial\gamma}{\partial x_\beta} \tag{A16}$$

from (A6) and (A7).

MODELE DE FERMETURE POUR LES ECOULEMENTS TURBULENTS ET INTERMITTENTS

Résumé—On développe un modèle de fermeture pour les écoulements turbulents "Shear flows", exclusivement à partir des moments conditionnels et du facteur d'intermittence. Le modèle contient les équations du facteur d'intermittence, des vitesses moyennes de la zone turbulente et de la zone non-turbulente, de l'énergie cinétique et du taux de dissipation pour la zone turbulente. Le modèle est appliqué à la prévision des jets plans et des couches limites et il donne des résultats satisfaisants pour le facteur d'intermittence et pour les moments du premier ordre.

Zusammenfassung—Ein Schliessungsmodell für turbulente Scherströmungen, das nur auf bedingten Momenten und Intermittenzfaktor beruht, wird entwickelt. Das Modell umfasst die Gleichumgen für Intermittenzfaktor, mittlere Geschwindigkeiten in turbulenter und nichtturbulenter Zone sowie kinetische Energie und Dissipationsrate in der turbulenten Zone. Das Modell zeigt bei Anwendung auf den ebenen Freistrahl und die Grenzschicht zufriedenstellende Resultate für Intermittenzfaktor und Momente erster Ordung.

ЗАМКНУТАЯ МОДЕЛЬ ДЛЯ ПЕРЕМЕЖАЮЩИХСЯ ТУРБУЛЕНТНЫХ ТЕЧЕНИЙ

Аннотация — Предложена замкнутая модель для турбулентного сдвигового течения, основанная только на условных моментах и коэффициенте перемежаемости. Модель содержит уравнения для коэффициента перемежаемости, средних скоростей в зонах турбулентного и нетурбулентного течения, а также кинетической энергии и скорости диссипации в зоне турбулентного течения. Модель используется для расчета плоских струй и пограничных слоев и дает удовлетворительные результаты для коэффициента перемежаемости и моментов второго порядка.